Slow-roll corrections to inflaton fluctuations on a brane

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Quantum fluctuations of an inflaton field, slow-rolling during inflation are coupled to metric fluctuations. In conventional four dimensional cosmology one can calculate the effect of scalar metric perturbations as slow-roll corrections to the evolution of a massless free field in de Sitter spacetime. This gives the well-known first-order corrections to the field perturbations after horizon-exit. If inflaton fluctuations on a four dimensional brane embedded in a five dimensional bulk spacetime are studied to first-order in slow-roll then we recover the usual conserved curvature perturbation on super-horizon scales. But on small scales, at high energies, we find that the coupling to the bulk metric perturbations cannot be neglected, leading to a modified amplitude of vacuum oscillations on small scales. This is a large effect which casts doubt on the reliability of the usual calculation of inflaton fluctuations on the brane neglecting their gravitational coupling.

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I. INTRODUCTION

Inflation is probably the simplest scenario for the origin of primordial fluctuations in our Universe [1]. Small scale vacuum fluctuations can be stretched to astrophysical scales by an period of accelerated expansion. Inflation provides a test of high-energy physics because the perturbations are generated from very short scales at high energies in the very early universe. These perturbations carry signatures from high energy physics, which can be tested by astronomical observations.

The slow-roll approximation [2] is a useful tool to study the fluctuations generated during inflation. If we can neglect the coupling to metric perturbations and the effective mass of the field then the perturbations are described by the fluctuations of a free scalar field in de Sitter spacetime. This gives the familiar result that the power spectrum of scalar field perturbations at horizon-crossing is given by $(H/2\pi)^2$. One can then calculate the comoving curvature perturbation which is conserved on super-horizon scales for adiabatic perturbations.

However inflaton perturbations will be coupled to gravity (metric perturbations) at first-order in the slow-roll parameters. In four-dimensional general relativity it is known how to consistently include linear metric perturbations by working in terms of the gauge invariant combination of scalar field and curvature perturbations, the so-called Mukhanov-Sasaki variable, which obeys a simple wave equation [3]. Gravitational effects are negligible at small scales and high energies, where perturbations can be normalised to the usual Bunch-Davies vacuum state. On large scales (super-horizon scales) the comoving curvature perturbation is conserved allowing one to relate observations of temperature anisotropies in the cosmic microwave background to high energy vacuum fluctuations during inflation. Exact solutions are known for the special case of power-law inflation in general relativity which generalise the de Sitter result and have been used to calculate first-order slow-roll corrections in more general inflation models [4].

In this paper, we develop a new way to derive slow-roll corrections based on a slow-roll expansion about de Sitter spacetime. In four-dimensional general relativity we show how to recover the usual first-order slow-roll corrections. Our method may be useful when one cannot derive an exact solution and the background spacetime is given as a perturbation about de Sitter spacetime.

We then apply our method to inflation in the brane world model. New ideas in the string theory suggest that our observable universe is a 4-dimensional hypersurface, or brane, in a higher dimensional bulk spacetime [5]. The simplest example of this model is the Randall-Sundrum model where there is a brane embedded in a 5-dimensional anti-de Sitter (AdS) spacetime [6]. An AdS spacetime has a characteristic curvature scale μ associated with the negative cosmological constant in the bulk. The spacetime shrinks exponentially away from the brane and this geometry effectively compactifies the 5-dimensional spacetime with the effective size μ^{-1}. On large length scales $L > \mu^{-1}$, 4-dimensional Einstein gravity is recovered, while on small scales, the gravity becomes 5-dimensional [7]. In the early universe when the Hubble horizon is smaller than μ^{-1}, we expect significant effects from higher dimensional bulk spacetime. Indeed, the Friedmann equation is modified from the conventional 4-dimensional theory for $H\mu \gg 1$ [8]. This modification of Friedmann can provide a novel model for inflation [9, 10, 11].

In Ref.[9], the amplitude of the curvature perturbation is calculated by taking into account the modification of the Friedmann equation. This work has been extended to include higher order corrections in slow-roll parameters [12] and the formula has been widely used to confront this model with the observations [13]. But to derive these formulae for the spectrum of the primordial curvature perturbations the effect of coupling to five-dimensional gravity

has been neglected and in particular it is assumed that the power spectrum of inflaton perturbations at horizon crossing is given by $(H/2\pi)^2$. This assumption is only valid to zeroth order in slow-roll parameters. At first order the inflaton perturbations will be coupled to metric perturbations. In the brane world, metric perturbations live in the 5-dimensional spacetime, and thus we must check if 5-dimensional effects change the result of conventional 4-dimensional theory. Especially, at small scales/high energies, the 5-dimensional effects could be large.

The first attempt to study the backreaction due to metric perturbations was made in Ref [14]. There, perturbations are solved perturbatively in slow-roll parameters. We should emphasize that this is the only possible way to perform the calculations analytically. If the background spacetime of the brane deviates from de Sitter spacetime, we cannot solve the bulk metric perturbations analytically. In contrast to four-dimensional general relativity, there are no other exact solutions known for the perturbation equations. Thus we must develop a new approach to calculate the effect of slow-roll corrections. In this paper, we extend earlier studies and investigate the backreaction due to higher-dimensional perturbations using a slow-roll expansion.

The structure of the rest of the paper is as follows. In section II, we describe our new approach to derive first order slow-roll corrections in a conventional 4-dimensional cosmology. In section III, we review an inflation model in the Randall-Sundrum brane-world driven by an inflaton field on the brane. In section IV, we derive the equations that govern the coupled system of inflaton fluctuations on the brane and metric perturbations in the bulk. In section V, the first order corrections to the inflaton fluctuations on the brane are solved. In section VI, we discuss the implications of our result for the brane world inflation model.

II. SLOW-ROLL EXPANSION OF SCALAR PERTURBATIONS IN 4D COSMOLOGY

A. Background spacetime

We consider an inflaton ϕ whose potential energy density $V(\phi)$ drives inflation. In the conventional 4-dimensional general relativity described by the metric

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \tag{1}$$

the Friedmann equation and the equation of motion for the homogeneous field, ϕ, are given by

$$H^2 = \frac{\kappa_4^2}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{2}$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}, \tag{3}$$

where $H = \dot{a}/a$, $\kappa_4 = 8\pi G_4$ and G_4 is the 4D gravitational coupling constant. A dot indicates a derivative with respect to cosmic time, t. Slow-roll parameters are defined by

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{4}$$

Slow-roll inflation is described by small values of ϵ and η.

B. Slow-roll corrections to inflaton fluctuations

The inhomogeneous inflaton fluctuation, $\delta\phi$, is coupled to the metric perturbations. In the Longitudinal gauge, the perturbed metric is written as

$$ds^2 = -(1+2\Psi)dt^2 + a(t)^2(1+2\Phi)\delta_{ij} dx^i dx^j. \tag{5}$$

The coupled equations for $\delta\phi$, Ψ and Φ can be simplified by using Mukhanov-Sasaki variable defined by [3]

$$u = a\left(\delta\phi - \frac{\dot{\phi}}{H}\Psi\right). \tag{6}$$

Expanding u by Fourier modes, the wave equation for u is given by

$$\frac{d^2 u_k}{d\tau^2} + \left(k^2 - \frac{1}{z}\frac{d^2 z}{d\tau^2}\right) u_k = 0, \tag{7}$$

where $z \equiv (a\dot{\phi})/H$ and τ is a conformal time defined as

$$\tau = \int \frac{dt}{a(t)}. \tag{8}$$

In the case of the slow-roll inflation, the mass term in Mukhanov-Sasaki equation (7) can be approximated as

$$\frac{1}{z}\frac{d^2 z}{d\tau^2} = \frac{1}{\tau^2}\left(2 + 6\epsilon - 3\eta + \mathcal{O}(\eta^2, \epsilon^2)\right), \tag{9}$$

up to first order of the slow-roll parameters. Then Eq. (7) can be expressed as

$$\frac{d^2 u_k}{d\tau^2} + \left(k^2 - \frac{1}{\tau^2}(2 + 6\epsilon - 3\eta)\right) u_k = 0. \tag{10}$$

Usually, the appropriately normalized solution with the correct asymptotic behavior at small scales is obtained by solving Eq. (10) directly as

$$u_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\pi/2}(-\tau)^{1/2} H_\nu^{(1)}(-k\tau), \tag{11}$$

where $\nu = 3/2 + 2\epsilon - \eta$ and $H_\nu^{(1)}$ is the Hankel function of the first kind of order ν. Here we assumed the Bunch-Davies vacuum state where perturbations stay in Minkowski vacuum at small scales. Equation (11) is an exact solution of the perturbation equation (10) only if the slow-roll parameters ϵ and η are constant. However their variation in a Hubble time is second-order and hence of higher-order in the slow-roll expansion. Thus we can take ϵ and η to be evaluated around the time of horizon-crossing.

We are interested in the asymptotic form of the solution well outside the horizon. Taking the limit $-k\tau \to 0$ yields the asymptotic form of u_k;

$$u_k \to e^{i(\nu-1/2)\pi/2} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\tau)^{-\nu+1/2}. \tag{12}$$

Expanding the gamma function in Eq. (12), we get

$$u_k \to e^{i(\nu-1/2)\pi/2} \Big\{ 1 + (2\epsilon - \eta)(2 - \gamma - \ln 2) \Big\} \frac{1}{\sqrt{2k}} (-k\tau)^{-1-2\epsilon+\eta}, \tag{13}$$

where we have used the formula for the poly-Gamma function

$$\psi(3/2) \equiv \frac{\Gamma'(3/2)}{\Gamma(3/2)} = 2 - \gamma - 2\ln 2, \tag{14}$$

where γ is an Euler number.

The quantity that is related to observables today is the the power spectrum of the curvature perturbation given by

$$\mathcal{P}_{\mathcal{R}}^{1/2}(k) = \sqrt{\frac{k^3}{2\pi^2}} \left| \frac{u_k}{z} \right|. \tag{15}$$

From Eqs. (9) and (10), it can be shown that, at large scale, the time dependences of u_k and z are the same, that is, $\mathcal{P}_{\mathcal{R}}^{1/2}$ is constant. Note that the constancy of $\mathcal{R}$ in the large scale limit does not depend on the slow-roll approximation, but holds for any adiabatic perturbation. Thus this comoving curvature perturbation can be related to the perturbation in the radiation density on large scales long after inflation has ended.

For the model with a monotonous potential, the following relation holds:

$$|z| = \frac{a|\dot{\phi}|}{H} = \frac{2}{\kappa_4^2} \frac{a}{H} \left| \frac{dH}{d\phi} \right|, \tag{16}$$

and conformal time can be evaluated up to the first order in slow-roll parameters as

$$\tau = -\frac{1}{aH}(1 + \epsilon). \tag{17}$$

Thus the power spectrum of the curvature perturbation is given by

$$\mathcal{P}_{\mathcal{R}}^{1/2} \;=\; [1 - (2C+1)\epsilon + C\eta] \frac{\kappa_4^2}{4\pi} \left\{ \frac{H^2}{|dH/d\phi|} \right\}_{k=aH}, \tag{18}$$

where $C = -2 + \ln 2 + \gamma \simeq -0.73$. The terms proportional to the slow-roll parameters are called Stewart-Lyth correction [4].

C. Perturbing about de Sitter spacetime

In this subsection, we reproduce the usual slow-roll corrections in a perturbative approach which does not require any exact solution of the perturbation equation other than that in a de Sitter spacetime. This will be more suited to extension to the case of brane-world gravity.

At zeroth order in slow-roll parameters, the spacetime is described by the de Sitter spacetime. Thus we can expand the spacetime from de Sitter spacetime. The scale factor is expanded as

$$a(t) = a^{(0)}(t) + a^{(1)}(t) + \mathcal{O}(\epsilon^2), \quad a^{(0)}(t) = \exp(Ht). \tag{19}$$

Accordingly, the Mukhanov-Sasaki variable is expanded as

$$u_k(\tau) = u_k^{(0)}(\tau) + u_k^{(1)}(\tau) + \mathcal{O}(\epsilon^2), \tag{20}$$

where $u_k^{(0)} \equiv a\delta\phi^{(0)}$ and $u_k^{(1)} \equiv a(\delta\phi^{(1)} - (\dot{\phi}/H)\Psi)$. Substituting Eq. (20) into Eq. (10), the zeroth order equation is given by

$$\frac{d^2 u_k^{(0)}}{d\tau^2} + \left(k^2 - \frac{2}{\tau^2}\right) u_k^{(0)} = 0. \tag{21}$$

Since we expect that the effects of the deviation from de-Sitter spacetime are insignificant at small scales, the form of $u_k^{(0)}$ is determined by demanding a Bunch-Davies vacuum

$$u_k^{(0)}(\tau) = A(-\tau)^{1/2} H^{(1)}_{3/2}(-k\tau), \tag{22}$$

where $A = (\sqrt{\pi}/2)e^{i\theta}$ and the phase θ is fixed so that $u_k(\tau) \to (1/\sqrt{2k})e^{-ik\tau}$.

Next, we must solve $u_k^{(1)}$ sourced by this zeroth order solution

$$\frac{d^2 u_k^{(1)}}{d\tau^2} + \left(k^2 - \frac{2}{\tau^2}\right) u_k^{(1)} - \frac{1}{\tau^2}(6\epsilon - 3\eta) u_k^{(0)} = 0. \tag{23}$$

If we impose the boundary conditions (i) $u_k^{(1)}(\tau)$ is negligible in the limit $\tau \to -\infty$, and (ii) $u_k^{(1)}(\tau)$ does not diverge faster than $u_k^{(0)}(\tau)$ in the limit $\tau \to 0$, then we find that the solution is given by,

$$\begin{aligned}
u_k^{(1)} &= C_1(-k\tau)^{1/2} J_{3/2}(-k\tau) + C_2(-k\tau)^{1/2} H^{(1)}_{3/2}(-k\tau), \\
C_1 &= \frac{\pi i}{2}(6\epsilon - 3\eta) A \int_{-\infty}^{\tau} d\tau' \frac{1}{\tau'} \left\{ H^{(1)}_{3/2}(-k\tau') \right\}^2, \\
C_2 &= \frac{-\pi i}{2}(6\epsilon - 3\eta) A \int_{-\infty}^{\tau} d\tau' \frac{1}{\tau'} H^{(1)}_{3/2}(-k\tau') J_{3/2}(-k\tau'),
\end{aligned} \tag{24}$$

where J_ν is the Bessel function of the order ν.

We take the limit $-k\tau \to 0$ and compare the asymptotic form with Eq. (13). Using the small arguments limit of the Bessel functions

$$J_{3/2}(x) \sim \left(\frac{x}{2}\right)^{3/2} \frac{1}{\Gamma(5/2)}, \quad H^{(1)}_{3/2}(x) \sim -i\frac{\Gamma(3/2)}{\pi} \left(\frac{2}{x}\right)^{3/2}, \tag{25}$$

we can show that the zeroth order Mukhanov-Sasaki variable approaches to

$$u_k^{(0)}(\tau) \to -4i\frac{\Gamma(\frac{3}{2})}{\pi} A(2k)^{-1/2}(-k\tau)^{-1}. \tag{26}$$

Next, we must evaluate the asymptotic form of the first order Mukhanov-Sasaki variable. Using Eq. (25), C_1 is evaluated as

$$C_1 \;\to\; 4i(2\epsilon - \eta) A \frac{\Gamma(3/2)^2}{\pi} \frac{1}{(-k\tau)^3}. \tag{27}$$

We should be careful in evaluating the asymptotic behavior of C_2 because sub-leading terms are comparable to the contribution from C_1. Using

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right), \quad J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{\cos x}{x}\right), \tag{28}$$

the integral for C_2 in Eq. (24) can be evaluated as

$$\int_{-\infty}^{\tau} d\tau' \frac{1}{\tau'} H^{(1)}_{3/2}(-k\tau') J_{3/2}(-k\tau') \simeq -\frac{2i}{3\pi}\mathrm{Ci}(-2k\tau) + \frac{14i}{9\pi}, \tag{29}$$

where Ci is the integrated cosine function defined as

$$Ci(x) \equiv -\int_x^{\infty} \frac{\cos t}{t} dt. \tag{30}$$

For small $-k\tau$, the integrated cosine function can be expressed as

$$Ci(-2k\tau) \to \gamma + \ln 2 + \ln(-k\tau). \tag{31}$$

Therefore, the asymptotic form of C_2 is given by

$$C_2 \;\to\; (2\epsilon - \eta) A \left(\gamma + \ln 2 + \ln(-k\tau) + \frac{7}{3}\right). \tag{32}$$

Then we obtain the asymptotic form of u_k for $-k\tau \to 0$ up to the first order in slow-roll parameters

$$u_k(\tau) \;\to\; (-i)e^{i\theta}\Big\{1 + (2\epsilon - \eta)(2 - \gamma - \ln 2)\Big\}\Big\{1 - (2\epsilon - \eta)\ln(-k\tau)\Big\}\frac{1}{\sqrt{2k}}(-k\tau)^{-1}, \tag{33}$$

where we have used the fact that $A = (\sqrt{\pi}/2)e^{i\theta}$. This should be compared with Eq. (13). There appears a logarithmic term which diverges for $-k\tau \to 0$. However, if we can renormalize this divergence by rewriting the logarithmic term as

$$1 - (2\epsilon - \eta)\ln(-k\tau) \simeq (-k\tau)^{-2\epsilon + \eta}. \tag{34}$$

we see that Eq. (33) is consistent with Eq. (13).

Indeed, the logarithmic divergence for $-k\tau \to 0$ in u_k does not show up in the spectrum of the curvature perturbation. In order to see this, we expand the curvature perturbation as

$$\mathcal{P}_{\mathcal{R}}^{1/2} = \{\mathcal{P}_{\mathcal{R}}^{1/2}\}^{(0)} + \{\mathcal{P}_{\mathcal{R}}^{1/2}\}^{(1)} + \mathcal{O}(\epsilon^2). \tag{35}$$

On the other hand, by the definition of the curvature perturbation (15), we can write the spectrum of curvature perturbation up to the first order in slow-roll parameters as

$$\mathcal{P}_{\mathcal{R}}^{1/2} \simeq \sqrt{\frac{k^3}{2\pi}}\left|\frac{u_k^{(0)}}{z^{(0)}} + \frac{u_k^{(0)}}{z^{(0)}}\left(\frac{u_k^{(1)}}{u_k^{(0)}} - \frac{z^{(1)}}{z^{(0)}}\right)\right|, \tag{36}$$

where we also expanded $z \equiv (a\dot{\phi})/H$ as

$$z = z^{(0)} + z^{(1)} + \mathcal{O}(\epsilon^2). \tag{37}$$

Since there is a difficulty to define the curvature perturbation in de Sitter spacetime, we concentrate on the ratio between the zeroth order and the first order of the curvature perturbation. By comparing Eq. (35) to (36), the ratio is given by

$$\frac{\{\mathcal{P}_{\mathcal{R}}^{1/2}\}^{(1)}}{\{\mathcal{P}_{\mathcal{R}}^{1/2}\}^{(0)}} = \frac{u_k^{(1)}}{u_k^{(0)}} - \frac{z^{(1)}}{z^{(0)}}. \tag{38}$$

In order to evaluate Eq. (38), we must obtain $z^{(0)}$ and $z^{(1)}$, that is, we must solve Eq. (9) perturbatively. Substituting Eq. (37) into Eq. (9), the equation for z at zeroth order is given by

$$\frac{d^2 z^{(0)}}{d\tau^2} = \frac{2}{\tau^2} z^{(0)}. \tag{39}$$

If we consider only the growing mode, the zeroth order solution for $z^{(1)}$ can be obtained as

$$z^{(0)} = B\tau^{-1}, \tag{40}$$

where B is an integration constant. This zeroth order solution gives a source term in the equation for z at first order;

$$\frac{d^2 z^{(1)}}{d\tau^2} = \frac{2}{\tau^2} z^{(1)} + \frac{(6\epsilon - 3\eta)}{\tau^2} z^{(0)}. \tag{41}$$

The growing mode solution for the first order $z^{(1)}$ is given by

$$z^{(1)} = -(2\epsilon - \eta) B\tau^{-1} \ln(-k\tau) + BD\tau^{-1}, \tag{42}$$

where D is another integration constant. Then we get

$$\frac{z^{(1)}}{z^{(0)}} = -(2\epsilon - \eta)\ln(-k\tau) + D. \tag{43}$$

This logarithmic divergence term exactly cancels the logarithmic divergence term in u_k;

$$\frac{u_k^{(1)}}{u_k^{(0)}} = (2 - \gamma - \ln 2 - \ln(-k\tau))(2\epsilon - \eta). \tag{44}$$

From Eqs. (43) and (44) we obtain

$$\frac{\{\mathcal{P}_{\mathcal{R}}^{1/2}\}^{(1)}}{\{\mathcal{P}_{\mathcal{R}}^{1/2}\}^{(0)}} = -C(2\epsilon - \eta) - D, \tag{45}$$

where $C = -2 + \ln 2 + \gamma \simeq -0.73$ is again a numerical constant. We cannot determine D in this approach, which comes from the difficulty to define curvature perturbation in pure de Sitter spacetime. However, we can still fix D as follows. Neglecting the logarithmic term, which is canceled by the contribution from u_k, the solution for z is written as

$$z = B(1 + D)\tau^{-1}. \tag{46}$$

This must be compared with the definition of z

$$z = \frac{a|\dot{\phi}|}{H} \sim -\frac{|\dot{\phi}|}{H^2}(1 + \epsilon)\tau^{-1}, \tag{47}$$

where the solution for a up to the first order was used. Then we can identify $B = -|\dot{\phi}|/H^2$ and $D = \epsilon$. Then Eq. (45) agrees with the Stewart-Lyth correction given by Eq. (18).

III. SLOW-ROLL INFLATION IN RANDALL-SUNDRUM BRANE WORLD

In this section, we apply our perturbative approach to the brane-world model. We consider the simplest version of brane-world inflation model based on the Randall-Sundrum model. We will consider a single brane embedded in a 5-dimensional AdS spacetime. We assume that the inflaton ϕ is confined to the brane while gravity can propagate in the whole 5-dimensional spacetime [9].

The 5-dimensional metric describing this model is given by [8]

$$ds^2 = dy^2 - N(y,t)^2 dt^2 + A(y,t)^2 \delta_{ij} dx^i dx^j, \tag{48}$$

where

$$A(y,t) = a(t)\left[\cosh\mu y - \left(1+\frac{\kappa^2\rho}{6\mu}\right)\sinh\mu y\right],$$
$$N(y,t) = \cosh\mu y - \left(1-\frac{\kappa^2\rho}{6\mu}(2+3w)\right)\sinh\mu y, \tag{49}$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{50}$$

and $w = P/\rho$. The brane is located at $y = 0$ and the inflaton is confined to this hypersurface. On the brane, the Friedmann equation and the equation of motion for the scalar field are given by

$$H^2 = \frac{\kappa_4^2}{3}\rho + \frac{\kappa^4}{36}\rho^2, \tag{51}$$
$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}, \tag{52}$$

where $\kappa_4^2 = \kappa^2\mu$, $\kappa^2 = 8\pi G_5$ and G_5 is 5D gravitational coupling. We can define slow-roll parameters in the same way as the conventional cosmology, Eq. (4).

Unfortunately, the background metric (48) is not in general a separable function with respect to y and t. Thus we cannot solve the metric perturbations analytically. In order to solve for the y-dependence of the bulk gravitons and to study the time-dependence of the perturbations on the brane, we will expand about the special case of a de Sitter spacetime on the brane. This corresponds to the background solution to zeroth order in a slow-roll expansion. For a de Sitter brane, AdS bulk gives a separable form for the bulk metric [15]:

$$ds^2 = dy^2 + N^2(y)\left[-dt^2 + a^2(t)\delta_{ij}dx^i dx^j\right], \tag{53}$$

where

$$a(t) = e^{Ht}, \tag{54}$$
$$N(y) = \frac{H}{\mu}\sinh\mu(y_{\rm h} - |y|),. \tag{55}$$

and $y = \pm y_{\rm h}$ are Cauchy horizons [15], with

$$y_{\rm h} = \frac{1}{\mu}\coth^{-1}\left(\sqrt{1+\left(\frac{H}{\mu}\right)^2}\right). \tag{56}$$

It is often useful to work in terms of the conformal bulk-coordinate $z = \int dy/N(y)$:

$$z = \mathrm{sgn}(y)H_o^{-1}\ln\left[\coth\tfrac{1}{2}\mu(y_{\rm h} - |y|)\right]. \tag{57}$$

The Cauchy horizon is now at $|z| = \infty$, and the brane is located at $z = \pm z_{\rm b}$, with

$$z_{\rm b} = \frac{1}{H}\sinh^{-1}\frac{H}{\mu}. \tag{58}$$

The line element, Eq. (53), becomes

$$ds^2 = N^2(z)\left[-dt^2 + dz^2 + e^{2Ht}d\vec{x}^{\,2}\right], \tag{59}$$

where

$$N(z) = \frac{H}{\mu\sinh(H|z|)}. \tag{60}$$

IV. EQUATIONS FOR BULK METRIC PERTURBATIONS AND INFLATON PERTURBATIONS ON THE BRANE

In this section, we derive the basic equations for the coupled Mukhanov-Sasaki variable on the brane and bulk metric perturbations following Ref.[14].

A. Master variable for perturbations in AdS bulk

In the background spacetime given by Eq. (59) bulk metric perturbations can be solved using the master variable [16, 17]. The perturbed metric is given by

$$ds^2 = N(z)^2\Big[(1+2A_{yy})dz^2 + 2A_y dt dz - (1+2A)dt^2 + a^2(1+2\mathcal{R})\delta_{ij}dx^i dx^j\Big]. \tag{61}$$

In the special case of a de Sitter brane in the AdS bulk, the metric variables are written by the master variable Ω as

$$A = -\frac{a^{-1}N^{-3}}{6}\left(2\Omega'' - 3\frac{N'}{N}\Omega' + \ddot{\Omega} - \mu^2 N^2\Omega\right), \tag{62}$$

$$A_y = a^{-1}N^{-3}\left(\dot{\Omega}' - \frac{N'}{N}\dot{\Omega}\right), \tag{63}$$

$$A_{yy} = \frac{a^{-1}N^{-3}}{6}\left(\Omega'' - 3\frac{N'}{N}\Omega' + 2\ddot{\Omega} + \mu^2 N^2\Omega\right), \tag{64}$$

$$R = \frac{a^{-1}N^{-3}}{6}\left(\Omega'' - \ddot{\Omega} - 2\mu^2 N^2\Omega\right). \tag{65}$$

From the perturbed 5-dimensional Einstein equation, we can derive the equation for Ω

$$\ddot{\Omega} - 3H\dot{\Omega} - \left(\Omega'' - 3\frac{N'}{N}\Omega'\right) + \frac{k^2}{a^2}\Omega - \mu^2 N^2\Omega = 0. \tag{66}$$

Solutions of the master equation can be separated into eigenmodes of the time-dependent equation on the brane and bulk mode equation:

$$\Omega(t,y;\vec{x}) = \int d^3\vec{k}\, dm\, \alpha_m(t) u_m(z) e^{i\vec{k}.\vec{x}}\,,$$

where

$$\ddot{\alpha}_m - 3H\dot{\alpha}_m + \left[m^2 + \frac{k^2}{a^2}\right]\alpha_m = 0\,, \tag{67}$$

$$u_m'' - 3\frac{N'}{N}u_m' + \mu^2 N^2 u_m + m^2 u_m = 0\,. \tag{68}$$

Note that the Hubble damping term $-3H\dot{\alpha}_m$ has the “wrong sign”, i.e., this is not the standard wave equation for a scalar field in four-dimensions.

If we write $\alpha_m = a^2\varphi_m$ and work in terms of the conformal time $\tau = -1/(aH)$, the time-dependent part of the wave equation (67) can be rewritten as

$$\frac{d^2\varphi_m}{d\tau^2} + \left[k^2 - \frac{2-(m^2/H^2)}{\tau^2}\right]\varphi_m = 0\,.$$

This is the same form of the time-dependent mode equation commonly given for a massive scalar field in de Sitter spacetime. The general solution is given by

$$\varphi_m(\eta;\vec{k}) = \sqrt{-k\tau}\, B_\nu(-k\tau)\,, \quad \nu^2 = \frac{9}{4} - \frac{m^2}{H^2}\,, \tag{69}$$

where B_ν is a linear combination of Bessel functions of order ν. The solutions oscillate at early-times/small-scales for all m, with an approximately constant amplitude while they remain within the de Sitter event horizon ($k \gg aH$). ‘Heavy modes’, with $m > \frac{3}{2}H$, continue to oscillate as they are stretched to super-horizon scales, but their amplitude rapidly decays away, $|u_m^2| \propto a^{-3}$. But for ‘light modes’ with $m < \frac{3}{2}H$, the perturbations become over-damped at late-times/large-scales ($k \ll aH$), and decay more slowly: $|u_m^2| \propto a^{2\nu-3}$.

B. Mukhanov-Sasaki equation on the brane

Now we introduce a scalar field fluctuation on the brane. We expand the scalar field perturbation in terms of slow-roll parameters;

$$\delta\phi = \delta\phi^{(0)} + \delta\phi^{(1)} + ... \tag{70}$$

The 0-th order of the scalar field fluctuation obeys the following equation of motion,

$$\delta\ddot{\phi}^{(0)} + 3H\delta\dot{\phi}^{(0)} + \frac{k^2}{a^2}\delta\phi^{(0)} = 0. \tag{71}$$

The metric perturbations are generated by the 0-th order fluctuation of the scalar field through the induced Einstein equations on the brane [18],

$$\begin{aligned} 3H\dot{\Psi} - 3H^2\Phi + \frac{k^2}{a^2}\Psi &= \frac{\kappa_{4,\mathrm{eff}}^2}{2}(\dot{\phi}\delta\dot{\phi}_0 + V'\delta\phi^{(0)}) + \frac{\kappa_4^2}{2}\delta\rho_E, & (72)\\ H\Phi - \dot{\Psi} &= \frac{\kappa_{4,\mathrm{eff}}^2}{2}\dot{\phi}\delta\phi^{(0)} - \frac{\kappa_4^2}{2}\delta q_E, & (73)\\ -\ddot{\Psi} - 3H\dot{\Psi} + H\dot{\Phi} + 3H^2\Phi - \frac{1}{3}\frac{k^2}{a^2}(\Psi+\Phi) &= \frac{\kappa_{4,\mathrm{eff}}^2}{2}(\dot{\phi}\delta\dot{\phi}_0 - V'\delta\phi^{(0)}) + \frac{\kappa_4^2}{6}\delta\rho_E, & (74)\\ -a^{-2}(\Psi+\Phi) &= \kappa_4^2\delta\pi_E, & (75) \end{aligned}$$

where

$$A(y=0,t) = \Phi(t), \quad R(y=0,t) = \Psi(t) \tag{76}$$

$$\begin{aligned} \kappa_4^2\delta\rho_E &= \frac{k^4a^{-5}}{3}\Omega & (77)\\ \kappa_4^2\delta q_E &= \frac{k^2a^{-3}}{3}\left(\dot{\Omega} - H\Omega\right), & (78)\\ \kappa_4^2\delta\pi_E &= \frac{a^{-3}}{2}\left(\ddot{\Omega} - H\Omega + \frac{k^2a^{-2}}{3}\Omega\right), & (79) \end{aligned}$$

and

$$\kappa_{4,\mathrm{eff}} = -\kappa_4 \left.\frac{N'}{N}\right|_{y=0}. \tag{80}$$

The contributions $\delta\rho_E, \delta q_E$ and $\delta\pi_E$ come from the projected 5D Weyl tensor and these describe the effect of the bulk gravitational perturbations [19]. The metric fluctuations in turn affect the dynamics of the first order scalar field perturbation

$$\ddot{\delta\phi}^{(1)} + 3H\dot{\delta\phi}^{(1)} + \frac{k^2}{a^2}\delta\phi^{(1)} = -V''\delta\phi^{(0)} - 3\dot{\phi}\dot{\Psi} + \dot{\phi}\dot{\Phi} - 2V'\Phi. \tag{81}$$

In order to evaluate the effect from metric perturbations, it is useful to use Mukhanov-Sasaki variable Q as in the conventional cosmology;

$$Q = \delta\phi - \frac{\dot{\phi}}{H}\Psi. \tag{82}$$

In terms of slow-roll expansion, we have $Q^{(0)} = \delta\phi^{(0)}$ and $Q^{(1)} = \delta\phi^{(1)} - (\dot{\phi}/H)\Psi$. Then using the induced Einstein equations, Eqs.(72), (74) and (75), we can derive the equation for $Q^{(1)}$;

$$\ddot{Q}^{(1)} + 3H\dot{Q}^{(1)} + \frac{k^2}{a^2}Q^{(1)} = -V''Q^{(0)} - 6\dot{H}Q^{(0)} + J, \tag{83}$$

where

$$J = -\frac{\kappa_4^2\dot{\phi}}{3H}\left(k^2\delta\pi_E + \delta\rho_E\right)$$
$$= -\frac{\dot{\phi}}{H}\frac{k^2a^{-3}}{6}\left(\ddot{\Omega} - H\dot{\Omega} + \frac{k^2}{a^2}\Omega\right). \tag{84}$$

The equation is the same as the standard 4-dimensional cosmology except for the term J, which describes the corrections from the 5-dimensional bulk perturbations. Because J contains the 5-dimensional quantity Ω we must solve the bulk equation for Ω to evaluate the effects.

C. Boundary condition for Ω

In order to solve Ω, we must specify the boundary condition for Ω. we rewrite the expressions of Φ and Ψ, Eq.(62) and (65), as [20]

$$\Psi = \frac{a^{-1}N^{-3}}{6}\left[3\frac{N'}{N}\mathcal{F} - 3H(\dot{\Omega} - H\Omega) - a^{-2}\Delta\Omega\right], \tag{85}$$

$$\Phi = \frac{a^{-1}N^{-3}}{6}\left[-3\frac{N'}{N}\mathcal{F} - 3\ddot{\Omega} + 6H\dot{\Omega} - 3H^2\Omega + 2a^{-2}\Delta\Omega\right]. \tag{86}$$

where

$$\mathcal{F} = \Omega' - \frac{N'}{N}\Omega. \tag{87}$$

Substituting these expressions into the induced Einstein equations (72)-(75), we obtain the equations written only by $\mathcal{F}$ and $\delta\phi^{(0)}$:

$$-3H\dot{\mathcal{F}} - k^2a^{-2}\mathcal{F} = \kappa^2 a(\dot{\phi}\dot{\delta\phi}^{(0)} + V'(\phi)\delta\phi^{(0)}), \tag{88}$$

$$\dot{\mathcal{F}} = \kappa^2 a\dot{\phi}\delta\phi^{(0)}, \tag{89}$$

$$\ddot{\mathcal{F}} + 2H\dot{\mathcal{F}} = \kappa^2 a(\dot{\phi}\dot{\delta\phi}^{(0)} - V'(\phi)\delta\phi^{(0)}). \tag{90}$$

These equations can be thought as the boundary conditions for Ω. Combining the junction conditions, Eqs.(88)-(90), we get an evolution equation for $\mathcal{F}$;

$$\ddot{\mathcal{F}} - \left(H + 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\dot{\mathcal{F}} + k^2a^{-2}\mathcal{F} = 0. \tag{91}$$

This is consistent with the equation for scalar field equation Eq.(71).

V. PERTURBATIVE SOLUTIONS

We must solve the coupled equations Eqs. (66) for Ω and Eq. (83) for Q. Introducing dimensionless quantities

$$Q(t) = Ha(t)^{-1}u(\tau), \quad \Omega(z,t) = \kappa^2\dot{\phi}H^{-1}\omega(z,\tau), \tag{92}$$

the coupled equations are written as

$$k^2\tau^2\left(\ddot{\omega} + \frac{4}{\tau}\dot{\omega} + k^2\omega\right) = \omega'' + 3\frac{\cosh Hz}{\sinh Hz}\omega' + \frac{1}{\sinh^2 Hz}\omega, \tag{93}$$

$$\dot{\mathcal{F}}_\omega = aH^2u, \quad \mathcal{F}_\omega = \left(\omega' + \frac{\cosh Hz}{\sinh Hz}\omega\right)_{z=z_b}, \tag{94}$$

$$\ddot{u} + k^2u - \frac{1}{\tau^2}(2 + 6\epsilon - 3\eta)u = J_u, \quad J_u = -\beta^2k^2\tau^2\left(\ddot{\omega} + \frac{2}{\tau}\dot{\omega} + k^2\omega\right), \tag{95}$$

where a dot denotes a derivative with respect to τ and

$$\beta^2 = \frac{\kappa^2 \dot{\phi}^2}{6H}. \tag{96}$$

At the leading order in slow-roll parameters, β^2 can be written as

$$\beta^2 = \frac{1}{3}\epsilon \frac{H}{\mu}\left(1+\left(\frac{H}{\mu}\right)^2\right)^{-1/2}. \tag{97}$$

Thus β^2 is essentially the slow-rolling parameter and it controls the strength of coupling between inflaton perturbation and gravitational perturbations in the bulk. We solve the coupled equations perturbatively in terms of small β^2.

A. Zeroth order solutions

At the zeroth order where $\beta^2 = 0$, the solution for u is given by

$$u^{(0)} = C_1(-k\tau)^{1/2} J_{-3/2}(-k\tau) + C_2(-k\tau)^{1/2} J_{3/2}(-k\tau). \tag{98}$$

Then $\mathcal{F}_\omega$ becomes

$$\mathcal{F}(\tau) = -C_1 H \sqrt{\frac{2}{\pi}} \frac{\cos(-k\tau)}{-k\tau} + C_2 H \sqrt{\frac{2}{\pi}} \frac{\sin(-k\tau)}{-k\tau}. \tag{99}$$

This gives the boundary condition for ω. The solution for ω in the bulk subject to this condition is obtained as [14]

$$\begin{aligned} \omega^{(0)}(z,\tau) &= -2C_1 \sum_{\ell=0}^{\infty}(-1)^\ell \left(2\ell+\frac{1}{2}\right) \frac{(\sinh Hz_b) Q_{2\ell}(\cosh Hz)}{\sinh Hz Q^1_{2\ell}(\cosh Hz_b)} (-k\tau)^{-3/2} J_{2\ell+1/2}(-k\tau), \\ &+2C_2 \sum_{\ell=0}^{\infty}(-1)^\ell \left(2\ell+\frac{3}{2}\right) \frac{\sinh Hz_b Q_{2\ell+1}(\cosh Hz)}{\sinh Hz Q^1_{2\ell+1}(\cosh Hz_b)} (-k\tau)^{-3/2} J_{2\ell+3/2}(-k\tau), \end{aligned} \tag{100}$$

where the identities

$$\cos(x) = \sqrt{2\pi} \sum_{\ell=0}^{\infty}(-1)^\ell \left(2\ell+\frac{1}{2}\right) x^{-\frac{1}{2}} J_{2\ell+1/2}(x), \tag{101}$$

$$\sin(x) = \sqrt{2\pi} \sum_{\ell=0}^{\infty}(-1)^\ell \left(2\ell+\frac{3}{2}\right) x^{-\frac{1}{2}} J_{2\ell+3/2}(x), \tag{102}$$

were used.

At large scales $-k\tau \to 0$, the dominant contribution comes from $\ell = 0$ mode. On the other hand, on small scales $-k\tau \to \infty$, all modes becomes comparable and we need to take into account an infinite ladder of the modes. This means that gravity becomes 5-dimensional at small scales.

In practice, we must approximate the infinite sum to proceed the calculations. We first check the identity Eqs. (101) and (102) to see if we can approximate the infinite summation by introducing a cut-off ℓ_c into the summation. From Fig. 1, we can see that if we increase the cut-off ℓ_c, the identity is satisfied for large $-k\eta$, i.e. on small scales. This implies that as long as we start from a finite time $-k\tau_i$, we can approximate the infinite ladder of the modes by introducing sufficiently large ℓ_c.

Fig. 2 shows the bulk solution for $\omega(z,t)$ with introducing sufficiently large cut-off ℓ_c. The solution is localized near the brane and decays towards the horizon $z \to \infty$. This is a bound state that is supported by an oscillation of the inflaton fluctuation on the brane. This kind of bound state generally appears in coupled brane and bulk oscillators [23]. A toy example is shown in Appendix. A key point here is that, in this case, the bound state is a summation of many different eigenstates of different eigenvalues (Eq. (100)). This fact becomes crucial in the analysis of the next order solution.

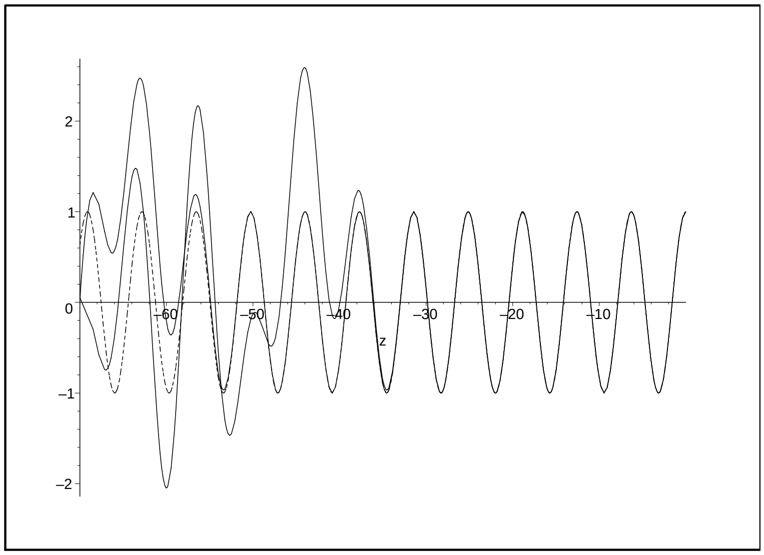

FIG. 1: Dotted lines shows $\cos(-z)$ and solid lines show the summation of Bessel functions with cut-off $\ell_c = 20$ and $\ell_c = 30$ respectively.

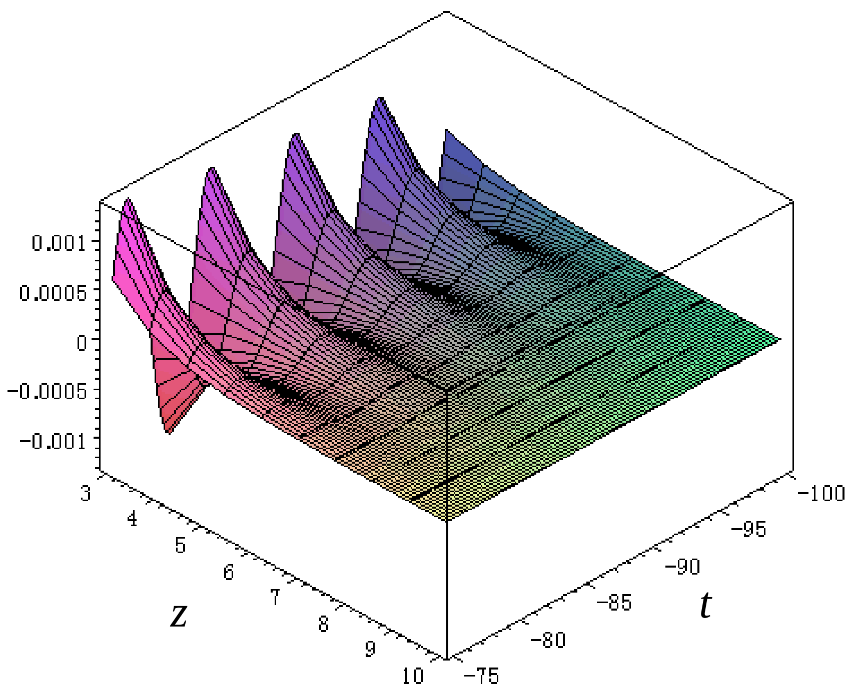

FIG. 2: The zeroth order solution for $\omega(z, t)$. A brane is located at $Hz = 3$.

B. First order solutions

Now it is possible to calculate the next order equation for $u^{(1)}$

$$\frac{d^2 u_k}{d\tau^2} + \left(k^2 - \frac{1}{\tau^2}(2 + 6\epsilon - 3\eta)\right) u_k = J_u, \tag{103}$$

where J_u describes the effect of the back reaction from the bulk perturbations. We can use the 0-th order solution to evaluate J_u as

$$J_u = \frac{2}{3}\epsilon k^2 C_1 \sum_{\ell=0}^{\infty}(-1)^\ell \left(2\ell+\frac{1}{2}\right) \triangle\,(2\ell; H\mu) \left(2\ell(2\ell-1)(-k\tau)^{-\frac{3}{2}} J_{2\ell+\frac{1}{2}} + 2(-k\tau)^{-\frac{1}{2}} J_{2\ell+\frac{3}{2}}(-k\tau)\right) \tag{104}$$

$$- \frac{2}{3}\epsilon k^2 C_2 \sum_{\ell=0}^{\infty}(-1)^\ell \left(2\ell+\frac{3}{2}\right) \triangle\,(2\ell+1; H\mu) \left(2\ell(2\ell+1)(-k\tau)^{-\frac{3}{2}} J_{2\ell+\frac{3}{2}} + 2(-k\tau)^{-\frac{1}{2}} J_{2\ell+\frac{5}{2}}(-k\tau)\right), \tag{105}$$

where

$$\triangle\,(n; H\mu) = \frac{H}{\mu}\left(1+\left(\frac{H}{\mu}\right)^2\right)^{-1/2} \frac{Q_n(\cosh Hz_b)}{Q_n^1(\cosh Hz_b)}. \tag{106}$$

The quantity $\triangle(n; H\mu)$ controls the amplitude of corrections to Mukhanov-Sasaki equations from the bulk over the change of the energy scales of the inflation.

In order to evaluate J_u, we need to introduce a cut-off in the summation at sufficiently large ℓ. Fig. 3 shows the behaviour of J_u against the change of the cut-off ℓ_c. A good feature here is that the behavior of J_u for small $-k\tau$ does not change even if we increase the cut-off. Thus we can reproduce the correct behaviour of J_u by a finite summation of modes as long as we are considering a finite time interval.

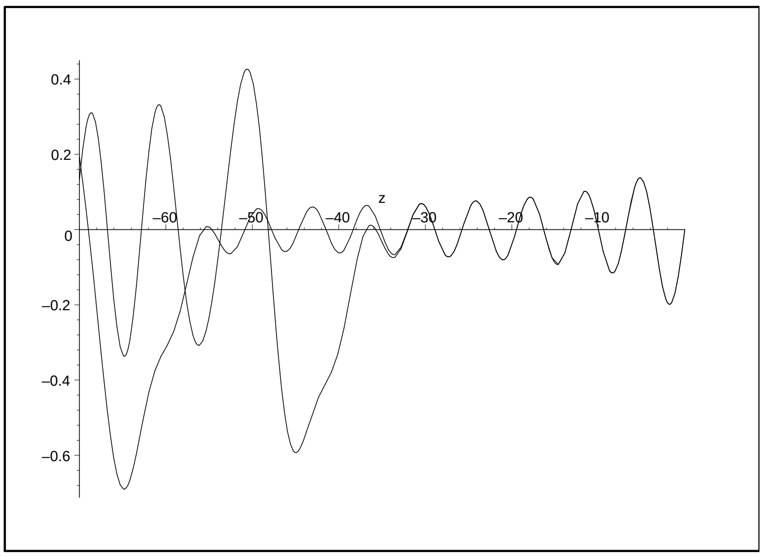

FIG. 3: Source term $J_u(-z)$ as a function of z with cut-off $\ell_c = 20$ and $\ell_c = 30$ respectively. Here we take $H\mu \gg 1$.

1. *Large scales*

On large scales $-k\tau \to 0$, $\ell = 0$ mode in C_1 mode dominates, which corresponds to a $m^2 = 2H^2$ mode. Thus we can approximate the infinite ladder of the modes by a single mode on super horizon scales. This indicates that, at large scales, gravity looks four-dimensional. Then we can easily show that J_u is suppressed for $-k\tau \to 0$ and the Mukhanov-Sasaki equation becomes completely the same as the conventional cosmology. Thus we can show the conservation of the curvature perturbation $\mathcal{R}$ on large scales in the same way as conventional cosmology [21].

2. *Small scales*

At low energies $H/\mu \ll 1$, $\triangle(n; H/\mu)$ can be approximated as

$$\triangle\,(n; H/\mu) = \left(\frac{H}{\mu}\right)^2 (\gamma + \psi(n+1) + \log(H/\mu) - \log 2)\,, \tag{107}$$

where we assumed n is not large. Thus, the source terms is well suppressed by the term $\triangle(n; H/\mu)$ at low energies. However, at sufficiently small scales, large ℓ modes become important and the approximation (107) does not hold. Then we could still get an effect on very sub-horizon scales ($k \gg \mu^{-1} \gg H$). In this case, we need to introduce a large cut-off in the summation of ℓ and it is technically difficult to perform a calculation.

At high energies, the amplitude of $\triangle(n; H\mu)$ becomes large as $H\mu$ becomes large, but, at sufficient high energies $H\mu \to \infty$, $\triangle(n; H\mu)$ becomes independent of $H\mu$ as seen from Fig.4. Indeed, we can obtain the asymptotic form of $\triangle(l; H/\mu)$ for $H/\mu \to \infty$ as

$$\triangle\,(l; H\mu) \to -\frac{1}{n+1}. \tag{108}$$

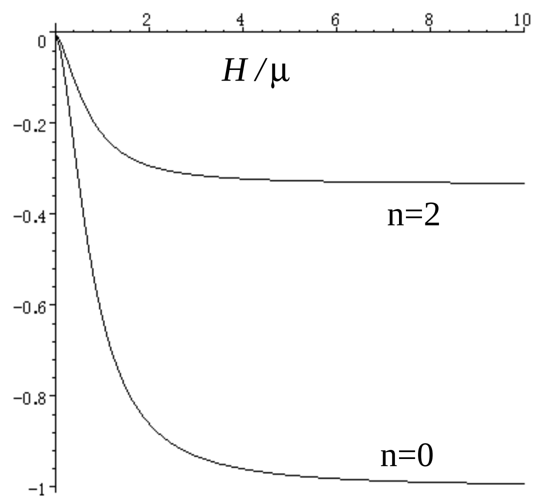

FIG. 4: $\triangle(n; H/\mu)$ as a function of H/μ.

In the following, we consider this limit. In this high energy limit, J_u is well fitted as

$$J_u \sim \frac{2\epsilon}{3} k^2 A \left[C_1 (-k\tau)^{-1/2} \cos(-k\tau + \varphi) - C_2 (-k\tau)^{-1/2} \sin(-k\tau + \varphi) \right], \tag{109}$$

between $-140 < k\tau < -40$ where $A = 0.4$ and $\varphi = 0.9$. Then, the equation of motion for the first order Mukhanov variable is given as

$$\frac{d^2 u_k^{(1)}}{d\tau^2} + \left(k^2 - \frac{2}{\tau^2} \right) u_k^{(1)} - \frac{1}{\tau^2}(6\epsilon - 3\eta) u_k^{(0)} - J_u(\tau) = 0. \tag{110}$$

By using the asymptotic behavior of the Bessel function at small scale (large $-k\tau$), the third term behaves like $(-k\tau)^{-2}\sin(-k\tau)$, while the forth term behaves as $(-k\tau)^{-1/2}\sin(-k\tau)$. Therefore, at least at small scales, the effect from the bulk metric perturbations dominates the effect from the standard corrections to the de Sitter geometry. Thus we will neglect the third term. The general solutions are given by the linear combination of $(-k\tau)^{1/2} J_{3/2}(-k\tau)$ and $(-k\tau)^{1/2} J_{-3/2}(-k\tau)$. By choosing the initial conditions so that $u_k(\tau_i) = u_k^{(0)}(\tau_i)$ at $\tau = \tau_i$, we find the following form of the solution,

$$u_k^{(1)} = D_1 (-k\tau)^{1/2} J_{\frac{3}{2}}(-k\tau) + D_2 (-k\tau)^{1/2} J_{-\frac{3}{2}}(-k\tau), \tag{111}$$

where D_1 and D_2 are given by

$$\begin{aligned} D_1 &= \frac{\pi}{2} \int_{k\tau_i}^{k\tau} d(k\tau')(-k\tau')^{\frac{1}{2}} J_{-\frac{3}{2}}(-k\tau') J_u(\tau'), \\ D_2 &= -\frac{\pi}{2} \int_{k\tau_i}^{k\tau} d(k\tau')(-k\tau')^{\frac{1}{2}} J_{\frac{3}{2}}(-k\tau') J_u(\tau'). \end{aligned} \tag{112}$$

For specifying the behavior of the first order Mukhanov variable, we must evaluate D_1 and D_2. Using the asymptotic form for Bessel functions at small scale, D_1 and D_2 are well approximated as

$$
\begin{aligned}
D_1 &\simeq -\frac{2\epsilon}{3}A\sqrt{\frac{\pi}{2}}\int_{k\tau_i}^{k\tau} d(k\tau')(-k\tau')^{-\frac{1}{2}}\sin(-k\tau')\left[C_1\cos(-k\tau'+\varphi)-C_2\sin(-k\tau'+\varphi)\right], \\
D_2 &\simeq \frac{2\epsilon}{3}A\sqrt{\frac{\pi}{2}}\int_{k\tau_i}^{k\tau} d(k\tau')(-k\tau')^{-\frac{1}{2}}\cos(-k\tau')\left[C_1\cos(-k\tau'+\varphi)-C_2\sin(-k\tau'+\varphi)\right]. \qquad (113)
\end{aligned}
$$

Then, on small scales, the first order solution is given by

$$
u_k^{(1)} \to ((F(\tau)-F(\tau_i))\cos(-k\tau)+(G(\tau)-G(\tau_i))\sin(-k\tau), \qquad (114)
$$

where

$$
\begin{aligned}
F(\tau) &= \frac{\epsilon A C_1\sqrt{\pi}}{3}\left[-S\left(\frac{2\sqrt{-k\tau}}{\sqrt{\pi}}\right)\cos\varphi-\left(C\left(\frac{2\sqrt{-k\tau}}{\sqrt{\pi}}\right)-\frac{2}{\sqrt{\pi}}\sqrt{-k\tau}\right)\sin\varphi\right] \\
&\quad+\frac{\epsilon A C_2\sqrt{\pi}}{3}\left[S\left(\frac{2\sqrt{-k\tau}}{\sqrt{\pi}}\right)\sin\varphi-\left(C\left(\frac{2\sqrt{-k\tau}}{\sqrt{\pi}}\right)-\frac{2}{\sqrt{\pi}}\sqrt{-k\tau}\right)\cos\varphi\right], \qquad (115) \\
G(\tau) &= \frac{\epsilon A C_1\sqrt{\pi}}{3}\left[\left(C\left(\frac{2\sqrt{-k\tau}}{\sqrt{\pi}}\right)+\frac{2}{\sqrt{\pi}}\sqrt{-k\tau}\right)\cos\varphi-S\left(\frac{2\sqrt{-k\tau}}{\sqrt{\pi}}\right)\sin\varphi\right] \\
&\quad+\frac{\epsilon A C_2\sqrt{\pi}}{3}\left[-\left(C\left(\frac{2\sqrt{-k\tau}}{\sqrt{\pi}}\right)+\frac{2}{\sqrt{\pi}}\sqrt{-k\tau}\right)\sin\varphi-S\left(\frac{2\sqrt{-k\tau}}{\sqrt{\pi}}\right)\cos\varphi\right], \qquad (116)
\end{aligned}
$$

where S and C are Fresnel functions.

We see that the first order perturbations grows like $\sqrt{-k\tau}-\sqrt{-k\tau_i}$. Then if we formally take the limit $-k\tau_i\to\infty$, the first order corrections diverge. Thus our perturbative approach breaks down. The amplitude of the zeroth order oscillation of inflaton fluctuations are significantly affected by the first order corrections.

We should take care in interpreting this result for the amplitude. In a toy model of a coupled boundary and bulk oscillators described in Appendix A, this change of amplitude due to the first order perturbations is merely caused by the breakdown of the perturbative expansion. In the toy model, the coupling to the bulk oscillator just changes the phase of the brane oscillator. In that case we can renormalize the first-order perturbation so that the first-order corrections appear only in the phase of the oscillations and do not have a large effect on the amplitude. However, in the case of inflaton fluctuations, we cannot do this kind of renormalization. This is due to the phase φ in the source term of the first order equation (see Appendix A). The phase originates from the fact that the zeroth order oscillation cannot be matched by a single bulk eigenmode with the same frequency as the brane oscillator and we need an infinite ladder of modes. Thus we can say that the effects on the amplitude from first order corrections are not artificial effects of our perturbative approach.

In conventional cosmology, the amplitude of inflaton oscillations u remains constant, so we can impose initial conditions on any scale far inside the horizon. However, in the brane world case, the coupling to the bulk metric perturbations changes the amplitude of the zeroth order inflaton oscillation u, so the effect crucially depends on the initial conditions. In general, classically, we can also impose arbitrary initial conditions for Ω. Indeed, it is always possible to add homogeneous solutions which satisfy the boundary condition given by

$$
\mathcal{F}=0. \qquad (117)
$$

Then we find an infinite tower of massive modes starting from $m^2=9H^2/4$. Arbitrary initial conditions for Ω can be satisfied by an appropriate summation of these massive modes. These massive modes also affect the evolution of inflaton fluctuations u [20].

We have tried to solve the coupled equations for inflaton fluctuations and master variable directly using a numerical method [22]. If we begin with the initial condition for ω given by Eq.(100), the numerical solution for u well agrees with our perturbative solutions as long as perturbations remain valid. We have also tried using different initial conditions for Ω and find that the effects on the amplitude of u depend on the initial conditions for ω in the bulk.

The initial conditions for u and ω must be determined by quantum theory on small scales. Thus we must quantise the coupled system of the inflaton fluctuations u and the master variable ω consistently. This is in contrast to the conventional cosmology where we can specify the vacuum for u by neglecting the gravitational effects far inside the horizon. This means that the assumption that the power spectrum of inflaton perturbations at horizon crossing is given by $(H/2\pi)^2$ could be invalid and we may have significant effects on the amplitude of perturbations from the backreaction due to the bulk metric perturbations.

VI. CONCLUSION

In this paper we have studied the effect of metric perturbations upon inflaton fluctuations during inflation, at first-order in slow-roll parameters ϵ and η, which describe the dimensionless slope and curvature of the potential. If we neglect the slope and curvature of the inflaton potential then we obtain the familiar results for free field fluctuations in de Sitter spacetime, with a scale invariant power spectrum on large (super-horizon) scales. We take this as our zeroth-order result in a slow-roll expansion.

In four-dimensional general relativity we were able to calculate corrections to the field evolution perturbatively to first-order in a slow-roll expansion, including linear metric perturbations. As far as we are aware this is the first time the slow-roll corrections have been calculated in the manner. We reproduce the familiar slow-roll corrections usually derived from Lyth and Stewart's exact solution to the linear perturbation equations in power-law inflation.

On a four-dimensional brane-world, embedded in a five-dimensional bulk, there are no exact solutions for cosmological perturbations (for a vacuum bulk described by Einstein gravity) except for the case of an exactly de Sitter brane. Thus the only way to calculate slow-roll corrections is perturbatively in a slow-roll expansion. We have calculated the leading order bulk metric perturbations sourced by the zeroth-order inflaton fluctuations on the brane. We find that inflaton fluctuations support an infinite tower of discrete bulk perturbations, with negative effective mass-squared.

Including the effect of the metric perturbations as an inhomogeneous source term in the wave equation for the first-order inflaton fluctuations we find that the effect of bulk metric perturbations becomes small on large scales, and we recover the usual result that the comoving curvature perturbation becomes constant outside the horizon.

However at small scales (or early times for a given comoving wavelength) the effect of bulk metric perturbations cannot be neglected. We are able to give an approximate solution for inflaton fluctuations at high energies and on sub-horizon scales using a truncated tower of bulk modes. This shows that the bulk metric perturbations change the amplitude of inflaton field fluctuations on the brane. By including a large number of bulk modes we can model this effect for many oscillations, but ultimately this change of amplitude becomes a large effect leading to a breakdown of our perturbative analysis.

It is not surprising in some ways that we see a large effect at small scales as these are high momentum modes which are expected to be strongly coupled to the bulk. Nonetheless this invalidates the usual assumption that gravitational effects are small far inside the cosmological horizon. It seems necessary to consistently solve for the coupled evolution of brane and bulk modes. We numerically tried to solve this problem and verified the validity of our perturbative approach as long as perturbations remain good. But it was also found that the change of the amplitude depends on the initial conditions for bulk metric perturbations. Detailed analysis of numerical solutions go beyond the scope of the present paper and they will be presented in a separate paper [22]. In order to give definite predictions for the amplitude of scalar perturbations in high energy inflation, we must specify the quantum vacuum state for coupled inflaton fluctuations and metric perturbations consistently and determine initial conditions. For this purpose, it would be useful to study the quantum theory of the toy model for a coupled bulk-brane oscillators in more details where we can consistently quantise a coupled system [23].

Our result implies the possibility that the assumption that the power spectrum of inflaton perturbations at horizon crossing on a brane is given by $(H/2\pi)^2$ could be invalid and we may have significant effects on the amplitude of perturbations from the backreaction due to the bulk metric perturbations.

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APPENDIX A: TOY MODEL FOR COUPLED BULK-BRANE SYSTEM

In this appendix, we present a simple toy model for a coupled brane and bulk oscillators. Let us consider a toy model for a brane field $q(t)$ and a bulk field ϕ in Minkowski bulk, which satisfy

$$\begin{aligned} \ddot{q} + \mu^2 q &= -\beta\phi, \\ \ddot{\phi} &= \phi'' - m^2\phi, \quad \phi'(y=0) = \frac{\beta}{2}q. \end{aligned} \tag{A1}$$

We solve the equations perturbatively in terms of small β. Without coupling, the zeroth order solution for q is given by

$$q^{(0)}(t) = C_1 \cos(\mu t) + C_2 \sin(\mu t). \tag{A2}$$

If we assume $m > \mu$, the 0-th order solution for ϕ is obtained as

$$\phi^{(0)} = -\frac{\beta}{2\sqrt{m^2-\mu^2}} \left(C_1 \cos(\mu t) + C_2 \sin(\mu t)\right) e^{-\sqrt{m^2-\mu^2} y}. \tag{A3}$$

Note that the bulk field has a negative effective mass-squared and decays towards $y \to \infty$. This is a normalizable bound state supported by an oscillation of $q(t)$ on the brane. The equation for the next order $q^{(1)}(t)$ is given by

$$\ddot{q}^{(1)} = -\mu^2 q^{(1)} + \frac{\beta^2}{2\sqrt{m^2-\mu^2}} \left(C_1 \cos \mu t + C_2 \sin \mu t\right). \tag{A4}$$

Including the zeroth order solution, the solution for $q(t)$ is given by

$$q^{(1)}(t) = C_1 \left(\cos \mu t + \frac{\beta^2}{4\mu\sqrt{m^2-\mu^2}} t \sin \mu t \right) + C_2 \left(\sin \mu t - \frac{\beta^2}{4\mu\sqrt{m^2-\mu^2}} t \cos \mu t \right). \tag{A5}$$

where we impose the initial condition so that $q(0) = q^{(0)}(0)$.

A problem is that the perturbation grows linearly in time. However, we need to be careful to interpret this growth of perturbations. In this toy model, we can easily find an exact solution. The corresponding exact solution becomes

$$\begin{aligned} q(t) &= C_1 \cos \left[\left(\mu - \frac{\beta^2}{4\mu\sqrt{m^2-\mu^2}} \right) t \right] + C_2 \sin \left[\left(\mu - \frac{\beta^2}{4\mu\sqrt{m^2-\mu^2}} \right) t \right], & \text{(A6)} \\ \phi(y,t) &= -\frac{\beta}{2\sqrt{m^2-\mu^2}} q(t) e^{-\sqrt{m^2-\mu^2} y}, & \text{(A7)} \end{aligned}$$

for $\beta \ll 1$. The effect of the coupling merely changes the frequency of the brane oscillator. The origin of the linear instability is that the naive expansion in terms of β is not efficient. Indeed, we can use

$$\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \sim \cos A - B \sin A, \\ \sin(A+B) &= \sin A \cos B + \sin B \cos A \sim \sin A + B \cos A, \end{aligned} \tag{A8}$$

for $B \ll 1$ and expand the exact solution into Eq.(A5). However, this perturbation breaks down for large t. A crucial difference of the inflaton fluctuations case from the toy model is that the source term for the first order equation for q contains a phase φ (compare Eqs. (109) and (110) to Eq. (A4)). Then a perturbative solution cannot be written into the form like Eq. (A6) using Eq. (A8). This indicates that there could be a modification of the amplitude as well as the phase shift. The phase φ is originated from the fact that the brane oscillation cannot be matched by a single bound state (compare Eq. (100) and Eq. (A3)). Thus this is an essential difference between the toy model and the inflaton fluctuations case.

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